LETTER TO THE EDITOR

Rossby-Haurwitz waves of a slowly and differentially rotating fluid shell

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Abstract. Recent studies have raised doubts about the occurrence of r modes in Newtonian stars with a large degree of differential rotation. To assess the validity of this conjecture we have solved the eigenvalue problem for Rossby-Haurwitz waves (the analogues of r waves on a thin-shell) in the presence of differential rotation. The results obtained indicate that the eigenvalue problem is never singular and that, at least for the case of a thin-shell, the analogues of r modes can be found for arbitrarily large degrees of differential rotation. This work clarifies the puzzling results obtained in calculations of differentially rotating axi-symmetric Newtonian stars.

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1. Introduction

The investigation of r modes in rotating stars has seen a renewed interest in recent years since they were shown to be unstable to the emission of gravitational waves (Andersson 1998, Friedman and Morsink 1998). In the last three years a number of papers have outlined the basic properties of the r-mode oscillations, their role in triggering and feeding the instability, and the importance of this instability in the emission of gravitational waves as well as in the slowing down of rapidly rotating, hot neutron stars (see Andersson and Kokkotas 2001 and Friedman and Lockitch 2001 for some recent reviews). Recent studies, on the other hand, have focussed on more intricate but also more realistic aspects of the instability. Such aspects include the interaction of the instability with magnetic fields (Spruit 1999, Rezzolla et al. 2000), nonlinear and secular effects (Rezzolla et al. 2001a, 2001b; Stergioulas and Font 2001; Lindblom et al. 2001), the presence of differential rotation in Newtonian stars (Karino et al. 2001, hereafter KYE), of a continuous frequency spectrum in the slow-rotation approximation of uniformly rotating relativistic stars (Kojima and Hosonuma 1999) and the lack of discrete, physically plausible mode solutions within the continuous part of the spectrum (Ruoff and Kokkotas 2001). We here pay attention to one of these more "subtle" aspects and focus on the role played by differential rotation in Newtonian stars. In particular, we address the question of whether a large degree of differential rotation in slowly rotating stars turns the eigenvalue problem for r modes into a singular eigenvalue problem, thus preventing the existence of r modes. This is an important question because of the possible observation of a singular problem in r-mode oscillations that has emerged from recent calculations of differentially rotating axi-symmetric Newtonian stars (KYE). More intriguingly, similar evidence is converging also from different approaches such as those looking at purely axial r modes in rotating non-isentropic relativistic stars in the slow-rotation

approximation (Yoshida 2001, Yoshida and Futamase 2001; Ruoff and Kokkotas, 2001a, 2001b) although a regularization may be possible in some cases (Lockitch and Andersson 2001).

Part of the subtlety introduced by differential rotation in the eigenvalue problem is due to the fact that, above a certain degree of differential rotation, the eigenvalue equations become very stiff as a result of large radial and polar gradients of the rotational angular frequency. Under these circumstances, the numerical solution requires increasingly high accuracy and computational costs. To circumvent this problem, we have here resorted to a simpler model based on a differentially rotating, thin shell. A thin-shell model has been adopted also by Levin and Ushomirsky (2001) to investigate the effect of electromagnetic radiation-reaction for r modes on a uniformly rotating shell. This approach replaces the set of partial differential equations of a multidimensional star model with an ordinary differential equation which can be solved to much higher accuracy and with modest computational costs. Although much simpler to solve, the r-mode eigenvalue problem for a thin shell incorporates many of the mathematical properties of the corresponding eigenvalue problems for multidimensional Newtonian stars or for slowly-rotating relativistic stars. As will become clear in the following Sections, adopting a thin-shell model has been very valuable for gaining insight into the behaviour of r modes in differentially rotating fluids.

2. Basic Equations

The thin-shell model adopted here is based on the following simplifying assumptions: *(i)* the background star is slowly rotating and we will omit terms in the hydrodynamical equations of $\mathcal{O}(\Omega^2)$; *(ii)* the fluid is incompressible and inviscid; *(iii)* the shell is spherical with radius R for every rate of differential rotation; *(iv)* the fluid motion is constrained on the shell, that is, we have no radial component of velocity.

In the inertial (nonrotating) frame the fluid velocity $\vec{v}$ is the composition of the unperturbed velocity of the differentially rotating shell $\vec{v}_{_0}$, where, in spherical polar coordinates, $v^i_{_0} = (0, 0, \Omega(\theta) R \sin\theta)$, with the velocity perturbation $\delta\vec{u}$ which has only tangential components, i.e. $\delta u^i = (0, u^\theta, u^\phi)$. As a result, the components of the fluid velocity can be written as

$$v^i = (0, u^\theta, \Omega R \sin\theta + u^\phi), \tag{1}$$

In the simple model considered here, the fluid motion in the shell is fully described by the Euler and continuity equations. The latter, in particular, is trivially satisfied by the background motion and assumes the following form in its perturbed part

$$\partial_\theta (u^\theta \sin\theta) + \partial_\phi u^\phi = 0\ . \tag{2}$$

As customary with this type of problem, it is here convenient to introduce the fluid vorticity $\vec{\xi} \equiv \frac{1}{2}\nabla \times \vec{v}$, whose radial component has the form

$$\xi^r = \Omega\cos\theta + \frac{1}{2}\frac{d\Omega}{d\theta}\sin\theta + \frac{1}{2r\sin\theta}\left[\partial_\theta(u^\phi\sin\theta) - \partial_\phi u^\theta\right]\ . \tag{3}$$

It is a simple exercise to show that the Euler equation can then be cast into a conservation equation for the vorticity $\partial_t\vec{\xi} + \nabla \times (\vec{\xi} \times \vec{v}) = 0$, whose radial component is

$$\left(\partial_t + \frac{v^\theta}{r}\partial_\theta + \frac{v^\phi}{r\sin\theta}\partial_\phi\right)\xi^r = 0\ . \tag{4}$$

Because we are interested in the harmonic modes, we assume that the tangential components of the perturbation velocity can be written as $u^\theta, u^\phi \sim \exp(-i\sigma t + im\phi)$, so that the linearized version of equation (4) is

$$-i(\sigma - m\Omega)\left[\frac{d}{d\theta}(u^\phi \sin\theta) - imu^\theta\right] + 2u^\theta \sin\theta \frac{d}{d\theta}\left(\Omega\cos\theta + \frac{1}{2}\frac{d\Omega}{d\theta}\sin\theta\right) = 0\,. \tag{5}$$

We can now use the continuity equation (2) to rewrite the radial component of the vorticity conservation equation as

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\chi}{d\mu}\right] - \frac{m^2}{1-\mu^2}\chi - \frac{2m\Omega}{\sigma - m\Omega}\left[1 + \left(\frac{2\mu}{\Omega}\right)\frac{d\Omega}{d\mu} - \left(\frac{1-\mu^2}{2\Omega}\right)\frac{d^2\Omega}{d\mu^2}\right]\chi = 0\,, \tag{6}$$

where we have introduced the new coordinate $\mu \equiv \cos\theta$ and where $\chi \equiv u^\theta \sin\theta$. Together with regular boundary conditions at the northern and southern poles ($\mu = \pm 1$), equation (6) accounts for the eigenvalue problem of the fluid normal modes on the shell. In the case of a uniformly rotating shell, $d\Omega/d\mu = 0 = d^2\Omega/d\mu^2$, and equation (6), first derived by Haurwitz in 1940 (see also Stewartson and Rickard, 1969), reduces to Legendre's equation whose regular solution is $\chi = P_l^m(\mu)$, with $P_l^m(\mu)$ being the associated Legendre functions. Note that in the case of uniform rotation the eigenfrequencies obey the well known dispersion relation for r modes

$$\sigma = m\Omega - \frac{2m\Omega}{l(l+1)}\,, \tag{7}$$

and this justifies calling these modes Rossby-Haurwitz waves. For simplicity, however, hereafter we will refer to them as r modes.

The solution of equation (6) in the case of a differentially rotating shell depends on the law of differential rotation chosen and on its first and second polar derivatives. The choice of a law of differential rotation is, in this sense, somewhat arbitrary but we have here followed previous work on the subject (Eriguchi and Müller 1985, KYE) and modeled the differential rotation through either a "$j = \mathrm{const.}$" law

$$\Omega = \frac{A_j^2 + 1}{A_j^2 + 1 - \mu^2}\Omega_{\mathrm{E}}\,, \tag{8}$$

or a "$v = \mathrm{const.}$" law‡

$$\Omega = \frac{A_v + 1}{A_v + \sqrt{1-\mu^2}}\Omega_{\mathrm{E}}\,. \tag{9}$$

In both cases Ω_{E} is the angular velocity at the equator ($\mu = 0$) and the parameter $A_{j,v} > 0$ accounts for the degree of differential rotation so that uniform rotation (i.e. $\Omega/\Omega_{\mathrm{E}} = \mathrm{const.}$) is reached for $A_{j,v} \to \infty$. Note that the use of a law of differential rotation is physically plausible as long as such a law does not violate Rayleigh's stability criterion for rotating inviscid fluids: $d\left(\varpi^2\Omega\right)^2/d\varpi > 0$, where ϖ is the cylindrical radial coordinate. It is straightforward to check that both expressions (8) and (9) satisfy Rayleigh's criterion for all values of $A_{j,v} > 0$.

The solution of equation (6) is considerably simpler than the solution of the corresponding set of partial differential equations describing normal modes of a differentially rotating axi-symmetric star and discussed by KYE. Nevertheless, most of the mathematical properties that are found in the solution of the system of partial differential equations can

‡ These differential laws take their names from the fact that through the variation of the parameter $A_{j,v}$ they represent families of rotation laws spanning the range between uniform rotation (for $A_{j,v} \to \infty$) and differential rotation with constant specific angular momentum or constant linear velocity (for $A_{j,v} \to 0$), respectively.

already be found in (6). A particularly important property of equation (6), which has been encountered also when dealing with r modes of differentially and rapidly rotating stars (KYE), is that it may become a singular eigenvalue problem at the angular position μ_s for which

$$\sigma - m\Omega(\mu_s) = 0 = \omega_{\rm ph} - \Omega(\mu_s) \ , \tag{10}$$

where $\omega_{\rm ph} \equiv \sigma/m$ is the phase velocity of the mode. The condition (10), which can be interpreted as the appearance of a *"corotation point"* (i.e. a point on the shell at which the perturbation pattern rotates at the same angular velocity as the background shell) has been interpreted by KYE as the cause impeding the numerical solution of the eigenvalue problem for a sufficiently large degree of differential rotation§. Interestingly, equation (6) offers close analogies also with the corresponding equation obtained in full General Relativity for a slowly and uniformly rotating relativistic star (Yoshida 2001, Yoshida and Futamase 2001; Ruoff and Kokkotas, 2001a, 2001b). The analogy is brought about by the fact that the corrections due to the relativistic dragging of inertial frames are mathematically similar to the corrections due to differential rotation and introduce a similar coefficient which could become singular for certain rates of rotation‖.

3. Strategy of the Numerical Solution

The numerical procedure adopted in the solution of equation (6) is straightforward and needs particular care only if a singular point should appear during the solution. In general, we can numerically integrate equation (6) as a two point boundary value problem with boundary conditions at both poles, $\mu = \pm 1$, being given by the requirement that the solution χ is regular there. The two solutions found are then matched at an arbitrary point $\mu_{\rm M}$ in the domain $(-1, 1)$ following the standard procedure in the solution of an eigenvalue problem. In particular, for a trial value of the eigenfrequencies we look for a zero of the Wronskian evaluated at $\mu_{\rm M}$, with different zeros representing sequences of different mode numbers. Once a zero is found, the eigenfrequency is used in equation (6) to calculate the corresponding eigenfunction. This procedure is repeated for different degrees of differential rotation so that a sequence is built describing the eigenfrequencies of the rotating shell from uniform rotation up to extreme differential rotation. The results of these calculations are discussed in detail in the following Section.

4. Numerical Results

This Section briefly presents the results of the numerical solution of equation (6) for the two laws of differential rotation discussed above.

4.1. $j = \mathrm{const.}$ *differential rotation*

We show in figure 1 the values of the phase velocity $\omega_{\rm ph}$ normalized to the value of the angular velocity at different latitudes on the shell. (This normalization is particularly convenient as it allows one to detect immediately whether corotation, corresponding to $\omega_{\rm ph}/\Omega(\mu) \to 1$, takes place or not.) As expected, in the limit of uniform rotation, the curves converge to $2/3$, the value given by the dispersion relation (7) for an $m = l = 2$, r mode. For progressively

§ KYE also noted that the difficulty in finding a convergent numerical solution was dependent on the rate of rotation of the background models, with rapidly rotating models providing solutions for comparably smaller values of $A_{j,v}$.

‖ Note that although equation (6) can in principle be singular, a global solution can still be found if series expansion techniques, such as the Frobenius method, are employed in the vicinity of the singular point μ_s.

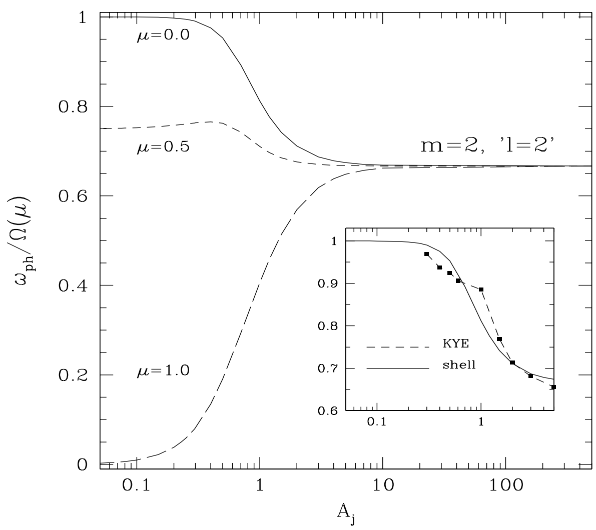

Figure 1. Behaviour of the normalized phase velocity $\omega_{\rm ph}/\Omega(\mu)$ for different values of the parameter A_j for a $j = {\rm const.}$ law of differential rotation. Different curves refer to the values of the eigenfrequencies at different latitudes on the shell. All of the curves refer to a mode which tends to the $m = l = 2$ r mode in the limit of uniform rotation (because of this we refer to it as an $m = 2$, '$l = 2$' mode). The small inset shows a comparison between the ratio $\omega_{\rm ph}/\Omega(\mu = 0)$ obtained with the present shell approach and the corresponding quantity obtained by KYE and indicated with filled squares.

smaller values of A_j, on the other hand, the curves split in response to the different angular velocities at different angular positions on the shell. Surprisingly, corotation is reached only at the equator (i.e. $\mu = 0$) and then only in the limit of $A \to 0$, which is not physically interesting.

This can be clearly seen in figure 2, where we show the deviation away from unity of the ratio $\omega_{\rm ph}/\Omega(\mu = 0)$. For both curves, which correspond in the limit of uniform rotation to an $m = l = 2$ and an $m = 2$, $l = 3$ mode respectively, the behaviour for small values of the parameter A is well fitted by a power law $\omega_{\rm ph}/\Omega(\mu = 0) \sim KA^n$, where $n \simeq 3.76$ for $m = 2$, '$l = 2$', and $n \simeq 53.4$ for $m = 2$, '$l = 3$' (K is here a positive constant). As a result, the deviation away from unity tends to zero only in the limit of $A \to 0$ and we therefore conclude that *no corotation* appears in these modes in the range of validity of a $j = {\rm const.}$ law of differential rotation.

The small inset in figure 1 shows a comparison between the ratio $\omega_{\rm ph}/\Omega(\mu = 0)$ obtained with the present shell approach and the corresponding quantity (indicated with filled squares) obtained by KYE with their numerical code for a slowly rotating stellar model with axis ratio 0.95. There is a rather good agreement, at least for the range of differential rotation rates in which the mode calculation was possible. Such an agreement brings confidence about the relevance of the results obtained with the shell model also for multidimensional star models.

When looking at the results of KYE, it becomes apparent how the behaviour of the eigenfrequencies might have suggested the appearance of a corotation point. As mentioned in the Introduction, when the degree of differential rotation increases past a certain threshold, very large radial and polar gradients appear in the equations for the eigenvalue problem. The solution of very stiff equations using finite difference techniques is a very difficult task and it is

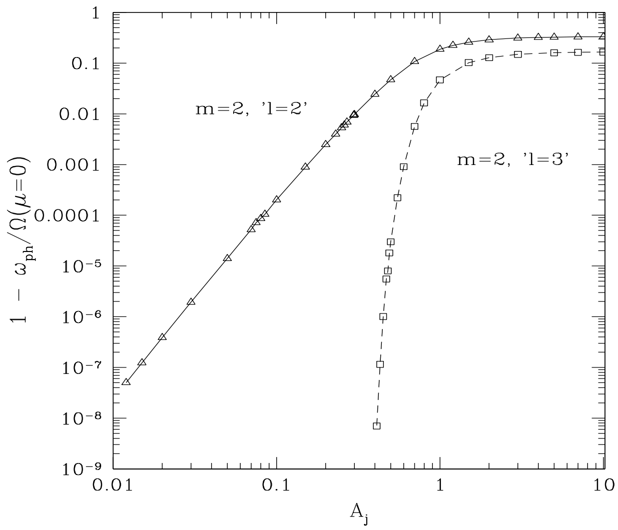

Figure 2. Deviation from unity of the ratio of the phase velocity to the local angular frequency at the equator (see figure 2). The small triangles and squares show the values of the computed eigenfrequencies.

therefore not surprising that KYE were not able to find convergent solutions to the eigenvalue problem for $A \lesssim 0.3$. By making use only of an ordinary differential equation, the shell approach bypasses this difficulty and provides an accurate solution for any value of A.

4.2. $v = \mathrm{const.}$ *differential rotation*

In the case of the $v = \mathrm{const.}$ law, the absence of a corotation point is even more evident. In figure 3 the eigenfrequency is plotted as in figure 1. There, we clearly see that the ratio $\omega_{\rm ph}/\Omega(\mu)$ is below unity everywhere on the shell. The comparison with the results by KYE, shown in the small inset of figure 3, is less good than the one seen for a $j = \mathrm{const.}$ differential rotation law but the overall behaviour is rather similar.

In addition to the differential rotation laws (8) and (9) we have also investigated a quadratic differential rotation law of the type $\Omega/\Omega_{\rm P} = 1 + (\mu^2 - 1)/B$, where $B > 0$ and uniform rotation is reached for $B \to \infty$. Also in this case, no evidence for corotation was found for the values of the parameter B satisfying the Rayleigh stability criterion (i.e $B > 2$).

5. Conclusion

We have used a thin-shell model to investigate the behaviour of Rossby-Haurwitz waves (the analogues of r waves on thin shells) in the presence of differential rotation. Our simplified approach replaces the set of partial differential equations for a multidimensional stellar model with a single ordinary differential equation that can be solved to arbitrary accuracy. The numerical solution of the eigenvalue problem has shown that there is no evidence of corotation, even for asymptotically large values of differential rotation. As a result, the eigenvalue problem for Rossby-Haurwitz waves never becomes singular, as had instead been suggested by recent calculations of differentially rotating axi-symmetric Newtonian stars. We have found that, for this simplified model, r modes can in principle coexist with arbitrarily

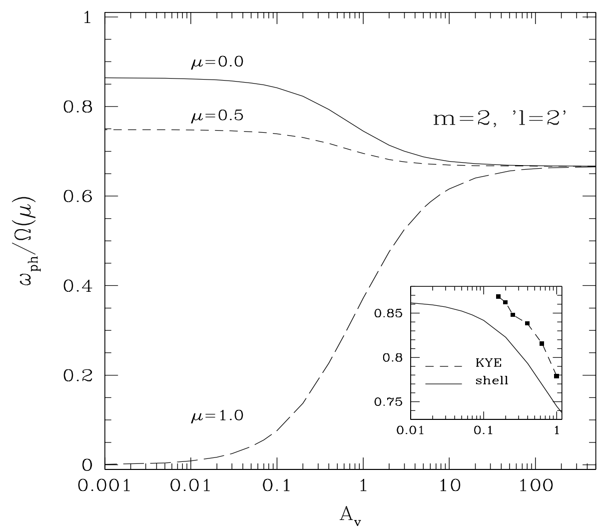

Figure 3. Same as in figure 3 but for a $v = \mathrm{const.}$ law of differential rotation.

large rates of differential rotation, the eigenfrequencies and eigenfunctions being suitably modified in response to the degree of differential rotation.

The relevance of the results found here for more realistic stellar models is not easy to assess, although the equations solved here show many of the mathematical features of the corresponding equations for Newtonian differentially rotating stars, or for slowly rotating relativistic stars.

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